A Construction of a Differential Graded Lie Algebra in the Category of Effective Homological Motives

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ABSTRACT

This text gives a construction of a differential graded Lie algebra in Nori's category of effective homological motives. In fact the construction works in more a general setting than that of an Abelian category. This allows us to give the rational homotopy Lie algebra of a 1-connected space a motivic structure. As a consequence the rational homotopy Lie algebra inherits a mixed Hodge structure and Galois module structure.

1. Introduction

The goal of this paper is to show that the rational homotopy Lie algebra of a variety (defined in Section 5.1) carries a motivic structure in the sense of Nori. This is formalized as follows. If $k \hookrightarrow \mathbb{C}$ is an embedding of a field k into the complex numbers then Nori (see [Br] and [Le]) defines an Abelian tensor category of effective homological motives over k, called IndEHM. Let $\mathcal{M} = \mathrm{Ch}(\mathrm{IndEHM})$ the Abelian category of chain complexes of motives. To say that X carries a motivic structure means that X is an object in $\mathcal{M}$ and that $H_*(X)$ is an object of EHM.

Nori uses his Basic Lemma [No] to construct a functor which assigns to any variety a complex of motives. A simpler case of this construction occurs when one considers the category of affine varieties. We denote this functor by $X \mapsto \mathrm{C}_*(X)$.

The restriction to affine varieties is not too severe thanks to "Jouanolou's Trick" (see [Jo]): For any quasi-projective variety X over a field k, there exists an affine variety X' over k and a morphism $X' \to X$ which is a Zariski locally trivial fibration with fibers isomorphic to $\mathbb{A}^n$. Since the affine fibers are contractible, X' is homotopy equivalent to X.

Nori also provides realization functors from IndEHM to the category of mixed Hodge structures and to the category of Galois representations. Let $\mathcal{M}$ be the category of chain complexes in IndEHM. There is a forgetful functor, ff from IndEHM to Abelian groups (and thus a forgetful functor from $\mathcal{M}$ to chain complexes of Abelian groups), such that

$$H_*(ff\mathrm{C}_*(X)) \cong H_*(X(\mathbb{C}), \mathbb{Z}),$$

where $H_*(X(\mathbb{C}), \mathbb{Z})$ denotes the singular homology of the topological space $X(\mathbb{C})$.

In this paper we provide a functor $\mathcal{P}_F$ which associates to any affine variety, X, a differential graded Lie algebra (d.g.l.), $\mathcal{P}_F(X)$ in $\mathcal{M}$, such that if the variety is simply connected then the homology of the d.g.l. computes the rational homotopy Lie algebra, that is

$$H_*(\mathcal{P}_F) \cong \pi_{*+1}(X) \otimes \mathbb{Q}$$

as graded Lie algebras. The known topological techniques for producing such a Lie algebra all use a coalgebra analogous to that of singular chains with the Alexander-Whitney map or the wedge product of forms.

This functor, $\mathcal{P}_F$, comes from an underlying combinatorial construction on geometric objects, and is perhaps the simplest Lie algebra one can construct from a simplicial object in an Abelian category which is representable (i.e. it can be extended to the category of finite sets with all morphisms, not just order preserving maps.) Applying the construction with singular chains to any topological space yields a new method of producing a d.g.l. which computes the rational homotopy Lie algebra.

This functorial construction has several applications. First, since Nori's category of motives has realization functors, our functor gives rise to a Galois structure on the homotopy Lie algebra after tensoring with $\mathbb{Q}_p$. Previously there was the mixed Hodge structure provided by Hain, but since he uses the graded commutative algebra structure of the deRham complex, his techniques were not easily adapted to the case of étale cohomology and Galois representations. Second, we pave the way for a motivic study of extensions arising from higher rational homotopy theory – analogous to Hain's work with extensions of mixed Hodge structures arising from the fundamental group [Ha1].

The serious difficulty in showing that the rational homotopy Lie algebra of a variety has a motivic structure reduces to the fact that although there is a motivic analog of the Eilenberg-Zilber map

$$\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y),$$

which is a quasi-isomorphism, there is no map in the other direction corresponding to the Alexander-Whitney map. Without the Alexander-Whitney map there is no coalgebra structure on motivic chains. It is unreasonable to expect a splitting of this map (like the Alexander-Whitney map) in a category of motives (otherwise the splitting forces non-zero extension classes to vanish). Thus $\mathrm{C}_*(X)$ does not have the structure of a coalgebra, in contrast to the topological case, where one has maps on singular chains

$$\mathrm{Sing}(X) \xrightarrow{\mathrm{Sing}(\Delta)} \mathrm{Sing}(X \times X) \xrightarrow{\mathrm{A-W}} \mathrm{Sing}(X) \otimes \mathrm{Sing}(X).$$

Put another way – it is clear how to take two cycles and produce a cycle on the product, but not vice versa. Thus much of the machinery of algebraic topology (in our case in particular anything related to differential graded Hopf algebras) can not be reproduced in the motivic category without much work.

Although we have no coalgebra, we still have a map $\mathrm{C}_*(X)^{\otimes n} \to \mathrm{C}_*(X^n)$ which is Σ_n equivariant. In some ways the key to this paper is that this Σ_n–equivariance is enough to get a Lie algebra and that we don't need (or have) a coalgebra structure.

One crucial point is that when V is a graded vector space, considering the tensor algebra $T(V)$ as a differential graded Hopf algebra, the primitive Lie algebra in $T(V)$ coincides with the free Lie algebra on V, $L(V)$ – and this can be obtained as the *image* of a projector defined in terms of the Σ_n action on $\mathrm{C}_*(X^n)$. This is crucial because defining this Lie algebra as primitives of a co-multiplication map would require a map $\mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ which does not exist since there is no map $\mathrm{C}_*(X \times X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$. The construction of $\mathcal{P}_F$ is very simple and combinatorial in nature. It has not been previously discussed before probably because: 1. Algebraic topologists would have no need to make this construction with the presence of the Alexander Whitney map (and thus the rich structure of Hopf algebras [MM]) 2. This construction lives outside of the simplicial category as we need to consider *all* maps between finite sets 3. Algebraic geometers such as [Ha] would make use of commutative algebras such as those arising from differential forms.

Several others have worked on related problems: Cushman's thesis, which produced a motivic structure on the group ring of the fundamental group completed at the augmentation ideal [Cu], provided a crucial idea that the geometric cobar complex could be used in Nori's category to compute the homology of the loop space. In the setting of mixed Hodge structures, building on the work of Chen, Hain showed that for any variety over $\mathbb{C}$, the Malcev completion of the fundamental

group and the rational homotopy Lie algebra (in the simply connected case) carry mixed Hodge structures [Ha1, Ha]. This work was further pursued by Wojtkowiak, who stressed the usefulness of using cosimplicial varieties [Wo].

The paper is organized as follows. In section 2, we lay out the categorical framework which we find most convenient to describe the d.g.l. and develop the algebraic identities in the category of Abelian groups which are used to deduce identities in a more general additive category $\mathcal{A}$. In section 3 we give the construction of the d.g.l. in $\mathcal{A}$. In section 4 we specialize to the category of effective homological motives. We summarize without proof many of the properties of Nori's construction of motives. This section reduces the problem to the case of singular chains, and thus to a problem in topology. In section 5 we compare this construction with the work of Hain to show that the forgetful functor applied to the d.g.l. in $\mathcal{M}$ gives a complex which computes the rational homotopy of a 1-connected variety.

The key mathematics in this work is in Section 3, in particular Section 3.2, where we show $\mathcal{P}_F$ is closed under its differential $F(f_n)$. Note that in diagram (63) ϕ_n is not f_n! Much of the insight needed to prove this theorem was obtained by constructing this ϕ_n "by hand". This was related to studying base-point dependency in the study of extensions of motives arising from the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ [De].

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2. Preliminaries

2.1 Conventions and Definitions

Let Fin be the category whose objects are finite sets and whose morphisms are all set maps (not just order preserving maps). Very contrary to the simplicial literature, we let $[n]$ denote a finite set with n elements labelled $1, 2, \ldots, n$.

The symmetric group on n letters, Σ_n, is the set of all bijections from $[n]$ to $[n]$. Since elements of Σ_n are maps, we think of Σ_n as acting on the left on $[n]$. Given a functor $G : \text{Fin}^{\text{opp}} \to \mathcal{C}$ to some category $\mathcal{C}$, the left Σ_n action on $[n]$ becomes a right Σ_n action on $G([n])$.

Often we will want to take linear combinations of maps of objects that have no Abelian group structure. To do this, given a category $\mathcal{C}$ we construct a new category $\mathcal{D}_\mathcal{C}$ (or just $\mathcal{D}$ if $\mathcal{C}$ is understood). The objects of $\mathcal{D}$ are the same as those of $\mathcal{C}$ but we define

$$\text{Hom}_\mathcal{D}(X, Y) = \mathbb{Z}\text{Hom}_\mathcal{C}(X, Y).$$

This gives an additive category where one can make sense of the notion of the composite of two maps being zero. We may also tensor with $\mathbb{Q}$ to give $\text{Hom}_\mathcal{D}$ the structure of a vector space. By definition an additive category has a zero object and a product structure.

A functor $G : \mathcal{C} \to \mathcal{C}'$ induces a functor (abusively called G) $G : \mathcal{D}_\mathcal{C} \to \mathcal{D}_{\mathcal{C}'}$. Furthermore, if $G : \mathcal{C} \to \mathcal{A}$ is a functor to an additive category, then the functor automatically extends uniquely to a functor $G : \mathcal{D}_\mathcal{C} \to \mathcal{A}$. We will be concerned with several functors in this paper:

2.2 Four Functors

We will be concerned with four functors on the category Fin.

   i) The functor represented by a set S.

      Let S be any set. S determines a functor (the "functor of $[n]$-valued points of S")

$$S : \text{Fin}^{\text{opp}} \to \text{Sets} \tag{1}$$

     given by $S(\bullet) = \text{Hom}_{\text{Fin}}(\bullet, S)$. Notice that $S([n]) = \text{Hom}_{\text{Sets}}([n], S)$ may be identified with the

product S^n (n-tuples of elements in S). A morphism of finite sets $f : [m] \to [n]$ induces a map $S^n \to S^m$ by pullback of coordinates.

Since $S(\bullet)$ is a functor, $S([n]) = S^n$ has a right action of Σ_n. For example if $n \geqslant 3$, $\sigma_{(123)} \in \Sigma_n$ acts on S^n:

$$(x_1, x_2, \ldots, x_{n-1}, x_n)\sigma = (x_2, x_3, x_1, \ldots, x_n)$$

We will apply this functor when S itself is a finite set.

ii) The functor represented by a graded $\mathbb{Z}$-module on a set.

A generalization of the functor above is to take the functor $\mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}(\bullet, S)$,

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}(\bullet, S) : \mathrm{Fin}^{\mathrm{opp}} \to \text{Abelian Groups} \tag{2}$$

Then $\mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([n], S)$ may be identified with $\mathbb{Z}[S]^{\otimes n}$, the n-th tensor power of the free module on S. If the elements S were to be given a grading then $\mathbb{Z}[S]^{\otimes n}$ would be a graded $\mathbb{Z}$ module. We could also tensor with a field k to obtain a vector space $k[S]^{\otimes n}$.

Σ_n acts on the n-th tensor power of $\mathbb{Z}[S]$ via the action on S. For example notice that the map $[2] \to [1]$ induces the linear map which on the basis S, sends $s_i \mapsto s_i \otimes s_i$.

The fact that V may be a graded vector space forces the symmetric group action to acquire a sign, e.g. $\sigma_{(12)}$ acts on $v \otimes w$ by

$$\sigma(v \otimes w) = (-1)^{|v|\,|w|} w \otimes v.$$

The graded rule from homological algebra is that anytime you pass one symbol X by another Y, you should multiply by $(-1)^{|X|\,|Y|}$, and this convention is a reflection of that one.

iii) The functor represented by a pair.

This is a generalization of the first functor in that first we generalize from sets to topological spaces and then from topological spaces to pairs. For a space X, throughout the rest of the paper we fix a base point $p \in X$. We will write X for the pair (X, p), unless otherwise indicated. Again we have

$$X^{\bullet} : \mathrm{Fin}^{\mathrm{opp}} \to \text{Pairs} \tag{3}$$

given by $[n] \mapsto X^n$ where X^n is the topologist's product $(X^n, p \times X^{n-1} \cup \ldots \cup X^{n-1} \times p)$ (which is homotopy equivalent to the smash $X^{\wedge n}$). Again, the morphisms are given by pull-back of coordinates. For example the map $[2] \to [1]$ induces the diagonal map $\Delta : X \to X \times X$.

iv) The functor $F : \mathrm{Fin}^{\mathrm{opp}} \to \mathcal{A}$.

Let our initial hypothesis be that $\mathcal{A}$ is an additive category with tensor product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ that satisfies the usual associativity and commutativity constraints (see [DMOS]). Also we assume there exists an identity object $1_{\mathcal{A}}$ for the tensor product. In the application to motives we will take $\mathcal{A} = \mathcal{M}$.

We also now assume the existence of a functor $F : \mathrm{Fin}^{\mathrm{opp}} \to \mathcal{A}$. Since $\mathcal{A}$ is additive, F extends to a functor from $\mathcal{D}_{\mathrm{Fin}}^{\mathrm{opp}}$ to $\mathcal{A}$. In application F will be F_X the functor determined by letting X be the pair of an affine variety and base point p and taking $F_X = \mathrm{C}_* \circ X^{\bullet} : \mathrm{Fin}^{\mathrm{opp}} \to \mathrm{Ch}(\mathcal{M})$.

2.3 The Disjoint Union and Product Structure

Given two finite sets $[p]$ and $[q]$ we can form their disjoint union to get a set with $p + q$ elements. That is to say, we have a functor

$$\amalg : \mathrm{Fin} \times \mathrm{Fin} \to \mathrm{Fin} \tag{4}$$

If $f \times f' : S \times S' \to T \times T'$ is a morphism in $\mathrm{Fin} \times \mathrm{Fin}$, then we write $f \amalg f' : S \amalg S' \to T \amalg T'$ for the functor $\amalg$ applied to $f \times f'$.

Let us consider the four functors mentioned above and their compatibility with $\amalg$.

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i) Sets. For a fixed set S, the following diagram obviously commutes.

$$
\begin{array}{ccc}
\mathrm{Fin} \times \mathrm{Fin} & \xrightarrow{\;\amalg\;} & \mathrm{Fin} \\
{\scriptstyle S \times S}\downarrow & & \downarrow{\scriptstyle S} \\
\mathrm{Sets} \times \mathrm{Sets} & \xrightarrow{\;\prod\;} & \mathrm{Sets}
\end{array}
\tag{5}
$$

That is, $\prod \circ (S \times S) = S \circ \amalg$.

ii) Free $\mathbb{Z}$ modules with a chosen basis S. The following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Fin} \times \mathrm{Fin} & \xrightarrow{\;\amalg\;} & \mathrm{Fin} \\
{\scriptstyle \mathrm{Hom}_{\mathcal{D}}(\bullet,S) \times \mathrm{Hom}_{\mathcal{D}}(\bullet,S)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{D}}(\bullet,S)} \\
\mathbb{Z}\mathrm{-mod} \times \mathbb{Z}\mathrm{-mod} & \xrightarrow{\;\otimes\;} & \mathbb{Z}\mathrm{-mod}
\end{array}
\tag{6}
$$

So we have $\otimes \circ (\mathrm{Hom}_{\mathcal{D}}(\bullet, S) \times \mathrm{Hom}_{\mathcal{D}}(\bullet, S)) = \mathrm{Hom}_{\mathcal{D}}(\bullet, S) \circ \amalg$.

iii) For the functor $X^\bullet$ we have the cartesian product completing the square:

$$
\begin{array}{ccc}
\mathrm{Fin} \times \mathrm{Fin} & \xrightarrow{\;\amalg\;} & \mathrm{Fin} \\
{\scriptstyle X^\bullet \times X^\bullet}\downarrow & & \downarrow{\scriptstyle X} \\
\mathrm{Pairs} \times \mathrm{Pairs} & \xrightarrow{\;\times\;} & \mathrm{Pairs}
\end{array}
\tag{7}
$$

and similarly $\times \circ (X^\bullet \times X^\bullet) = X^\bullet \circ \amalg$.

iv) For the functor $F : \mathrm{Fin}^{\mathrm{opp}} \to \mathcal{A}$, we do **not** require that

$$
\begin{array}{ccc}
\mathrm{Fin} \times \mathrm{Fin} & \xrightarrow{\;\amalg\;} & \mathrm{Fin} \\
{\scriptstyle F \times F}\downarrow & & \downarrow{\scriptstyle F} \\
\mathcal{A} \times \mathcal{A} & \xrightarrow{\;\otimes\;} & \mathcal{A}
\end{array}
\tag{8}
$$

commutes.

We weaken this and only require that we have a natural transformation of functors

$$
N : \otimes \circ (F \times F) \to F \circ \amalg.
\tag{9}
$$

In practice this will be Nori's motivic version of the Eilenberg–Zilber map.

Let us further assume that F satisfies the following axioms. Together with the axioms in Section 4.2, these are the axioms which Nori advocates as a replacement for E_∞ coalgebras. We have not explored the relationship with the work of Segal ([Se]) who proposed similar axioms in the topological setting.

For the natural transformation on pairs, for all $S, T \in \mathrm{Fin}$,

$$
\begin{array}{c}
\otimes \circ (F \times F) \xrightarrow{\;N\;} F \circ \amalg \\[4pt]
FS \otimes FT \xrightarrow{\;N(S,T)\;} F(S \amalg T)
\end{array}
\tag{10}
$$

satisfies

i) If $S = \emptyset$ then $F(S) = 1_{\mathcal{A}}$.

ii) For all $S \in \mathrm{Fin}$ we have both

$$
\begin{array}{c}
F\emptyset \otimes FS \xrightarrow{\;N(\emptyset,S)\;} FS \\[4pt]
FS \otimes F\emptyset \xrightarrow{\;N(S,\emptyset)\;} FS
\end{array}
\tag{11}
$$

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are canonical identifications in $\mathcal{A}$

iii) (commutativity) The switch map $\sigma : T \amalg S \to S \amalg T$ induces a map

$$F(\sigma) : F(S \amalg T) \to F(T \amalg S)$$

making the following diagram commute:

$$
\begin{array}{ccc}
FS \otimes FT & \xrightarrow{\ N(S,T)\ } & F(S \amalg T) \\
\downarrow & & \downarrow{\scriptstyle F(\sigma)} \\
FT \otimes FS & \xrightarrow{\ N(T,S)\ } & F(T \amalg S)
\end{array}
\tag{12}
$$

iv) (associativity) Let $R, S, T \in$ Fin then the following diagram commutes

$$
\begin{array}{ccc}
(FR \otimes FS) \otimes FT & \xrightarrow{\ N(R,S)\otimes 1\ } & F(R \amalg S) \otimes FT \\
\downarrow & & \searrow{\scriptstyle N(R\amalg S,T)} \\
& & F(R \amalg S \amalg T) \\
& \nearrow{\scriptstyle N(R,S\amalg T)} & \\
FR \otimes (FS \otimes FT) & \xrightarrow{\ 1\otimes N(S,T)\ } & FR \otimes F(S \amalg T)
\end{array}
\tag{13}
$$

PROPOSITION 2.1 COROLLARY TO THE AXIOMS.   i) $F([p]) \otimes F([q]) \xrightarrow{N([p],[q])} F([p+q])$ is $\Sigma_p \times \Sigma_q$-equivariant.

ii) $(N([n-1],[1])) \circ (N([n-2],[1]) \otimes [1]) \circ \cdots \circ (N([1],[1]) \otimes 1 \otimes \cdots \otimes 1) : F([1])^{\otimes n} \to F([n])$ is Σ_n – equivariant.

Proof. Both i and ii are consequences of N being a natural transformation.

$\square$

2.4 Useful Algebraic Identities

The goal of this section is to observe some identities in the group ring $\mathbb{Z}[\Sigma_n]$. The identities will be applied by considering

$$\mathbb{Z}[\Sigma_n] \hookrightarrow \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([n],[n]).
\tag{14}$$

All vector spaces are taken over a field of characteristic zero.

We begin by describing some properties of Lie algebras in terms of elements in $\mathbb{Z}[\Sigma_n]$.

For each $n \in \mathbb{N}$ define

$$s_n = (1 - \sigma_{(12)}) \cdots (1 - \sigma_{(12...n-1)})(1 - \sigma_{(12...n)}) \in \mathbb{Z}[\Sigma_n].
\tag{15}$$

If $n = 1$ let $s_n = 1$. If V is a vector space (concentrated in degree 0), then right multiplication by s_n above, R_{s_n}, acts on $V^{\otimes n}$ exactly as the Lie bracket:

$$
V^{\otimes n} \xrightarrow{\ \ R_{s_n}\ \ } V^{\otimes n}
\tag{16}
$$

$$v_1 \otimes \ldots \otimes v_n \longmapsto [\ldots [v_1, v_2]\ldots, v_n]$$

For each $n \in \mathbb{N}$ define $w_n \in \mathbb{Z}[\Sigma_n]$ by

$$w_n = (1 + \sigma_{(12)}) \ldots (1 + (-1)^{n-1}\sigma_{(123...n-1)}) \ldots (1 + (-1)^n \sigma_{(123...n)})
\tag{17}$$

and if $n = 1$, let $w_n = 1$. w_n is s_n twisted by the sign representation, $\epsilon : \Sigma_n \to \{\pm 1\}$:

$$w_n = \prod_{i=2}^{n} (1 - \epsilon(\sigma_{(12...i)})\sigma_{(12...i)}) \in \mathbb{Z}[\Sigma_n]. \tag{18}$$

If V is a vector space concentrated in odd degree, then w_n acts on the right by the Lie bracket:

$$R_{w_n}(v_1 \otimes \ldots \otimes v_n) = (v_1 \otimes \ldots \otimes v_n)w_n = [\ldots [v_1, v_2] \ldots, v_n] \tag{19}$$

Let $\mathcal{L}(V)$ denote the free Lie algebra on V. If V is graded then $\mathcal{L}(V)$ is a graded Lie algebra. We write $\mathcal{L}(V) = \bigoplus_{i=1}^{\infty} L_i(V)$ where L_i is the vector space of i-fold iterated brackets of elements of V.

Henceforth let V be concentrated in degree one, then $L_i(V) = V^{\otimes i}w_i$, so that $\mathcal{L}(V) = \bigoplus_{i=1}^{\infty} V^{\otimes i}w_i$. Let $T(V)$ be the tensor algebra on V. $T(V)$ can be identified with the universal enveloping algebra of the free Lie algebra on V, giving $T(V)$ the structure of a Hopf algebra (see [MM] and [Qu]). Then using the coalgebra structure on $T(V)$, we have that $\mathcal{L}(V)$ is the primitive Lie algebra of primitive elements of $T(V)$. Now the w_i assemble to give a linear map, $w : T(V) \to T(V)$, where w acts on the degree n terms of $T(V)$ as right multiplication by w_n. Thus the image of w is $\mathcal{L}(V)$. Where normally $\mathcal{L}(V)$ is thought of as the kernel of the reduced co-multiplication induced from $U(L) \to U(L) \otimes U(L)$ (i.e. the Lie algebra of primitives) we have identified $\mathcal{L}(V)$ with the image of $w : T(V) \to T(V)$.

Notice that :

$$s_n^2 = ns_n. \tag{20}$$

The analog for w_n is the following Proposition.

PROPOSITION 2.2.

$$w_n^2 = nw_n \tag{21}$$

Proof. The proof of the following Lemma is used in Quillen's proof of the Poincaré-Birkhoff-Witt theorem and is contained in his paper [Qu]:

LEMMA 2.3. *The map* $\rho : T(V) \to \mathcal{L}(V)$ *given by*

$$\rho(x_n \otimes x_{n-1} \otimes \ldots \otimes x_1) = \begin{cases} \frac{1}{n}\mathrm{ad}_{x_n}\mathrm{ad}_{x_{n-1}} \ldots \mathrm{ad}_{x_2}(x_1) & \text{if } n > 0, \\ 0 & \text{if } n = 0 \end{cases} \tag{22}$$

is a left inverse for the map $\mathcal{L}(V) \to T(V)$

ρ is almost the linear map given by right multiplication by $\frac{1}{n}w_n$, since ρ sends $v_1 \otimes \ldots \otimes v_n$ to $\frac{1}{n}[x_1, [x_2, \ldots [x_{n-1}, x_n] \ldots]$ while $\frac{1}{n}R_{w_n}$ sends $v_1 \otimes \ldots \otimes v_n$ to

$$[\ldots [x_1, x_2], x_3], \ldots, x_n].$$

Clearly $\frac{1}{n}R_{w_n}$ is also a left inverse for the map $\mathcal{L}(V) \to T(V)$.

Given $l \in \mathcal{L}(V)$, since $\mathcal{L}(V)$ is the free Lie algebra on a vector space V, l is a linear combination (after using the Jacobi identity) of terms of the form

$$[\ldots [x_1, x_2], x_3], \ldots x_n].$$

Considering this as an element of $T(V)$, l can be expressed as a linear combination of terms like $R_{w_n}(x_1 \otimes \ldots \otimes x_n)$. The Lemma states that applying $\frac{1}{n}R_{w_n}$ to these elements is the identity:

$$(x_1 \otimes \ldots \otimes x_n)w_n \circ \frac{1}{n}w_n = (x_1 \otimes \ldots \otimes x_n)w_n. \tag{23}$$

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Multiplying both sides by n gives $w_n^2 = nw_n$. $\square$

Notice that when we identify $\mathcal{L}(V)$ with $\mathrm{im}(w) \subset T(V)$, we have that when restricting w to degree n, $w^2 = nw$. Thus the following sequence is exact

$$0 \longrightarrow (R_{w_n} V^{\otimes n}) \overset{\subset}{\longrightarrow} V^{\otimes n} \overset{R_{n-w_n}}{\longrightarrow} V^{\otimes n}. \tag{24}$$

This may be rewritten; in degree n we have:

$$0 \longrightarrow L_n(V) \overset{i}{\hookrightarrow} (TV)_n \overset{R_{n-w_n}}{\longrightarrow} (TV)_n \tag{25}$$

Thus $R_{w_n} : (TV)_n \twoheadrightarrow L_n(V)$ is a splitting for $L_n(V) \hookrightarrow (TV)_n$.

Let us recall some definitions. Let R be a graded algebra over a field. A graded algebra derivation of degree $|d|$ is a linear map $d : R \to R$ which satisfies

$$d(u \cdot v) = du \cdot v + (-1)^{|u||d|} u \cdot dv \tag{26}$$

This implies that when we consider R, with the bracket given by $[u, v] = uv - (-1)^{|u||v|} vu$, d is a Lie derivation:

$$d[u, v] = [du, v] + (-1)^{|u||d|}[u, dv]. \tag{27}$$

Now fix a basis $v_1, \ldots v_m$ for a vector space V concentrated in degree 1. We will consider the set $S = \{v_1, \ldots, v_m\}$ and apply the functor $\mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([n], S) \otimes k$ in the next section. Let $Dx = nx$ if $x \in L_n(V)$. Then a straightforward computation shows that $D : \mathcal{L}(V) \to \mathcal{L}(V)$ is a derivation of degree 0.

For $w : TV \to TV$ defined above, we see that $D = w$ when restricted to $\mathcal{L}(V)$. That is w is *not* an algebra derivation of TV, though it is the Lie algebra derivation given by D on $\mathcal{L}(V) \subset TV$.

Before defining our second useful derivation we need the following

LEMMA 2.4. a. *Let V be a graded vector space concentrated in degree 1. Suppose that $\alpha : V \to TV$ is a linear map "of pure degree k," (i.e. so that $V \to V^{\otimes k+1}$). Then α extends to a graded derivation $d : TV \to TV$.*

  b. *If $\alpha(V) \subset \mathcal{L}(V)$, then d restricts to a Lie derivation, and thus $d(\mathcal{L}(V)) \subset \mathcal{L}(V)$.*

Proof. a. To give a derivation on an algebra we need only define d on generators. Since TV is the free algebra on V, d extends to products by

$$d(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n} (-1)^{(i-1)k} v_1 \otimes \ldots \otimes v_{i-1} \otimes \alpha(v_i) \otimes v_{i+1} \otimes \ldots \otimes v_n. \tag{28}$$

b. An algebra derivation gives a Lie algebra derivation as in (27). Now d takes the generators of $\mathcal{L}(V)$ to $\mathcal{L}(V)$ by hypothesis. To show that $d(\mathcal{L}(V)) \subset \mathcal{L}(V)$ proceed by induction on bracket length: if

$$d(\oplus_{i=1}^{k} L_i(V)) \subset \mathcal{L}(V) \tag{29}$$

then write (using Jacobi) any element $x \in L_{k+1}$ as a linear combination of elements of the form $[y, z]$ where y and z are of bracket length at most k. Then applying formula (27) we see $d[y, z]$ is written as a sum of $[dy, z]$ and $[y, dz]$ which are both in $\mathcal{L}(V)$ by induction. $\square$

We now introduce the second derivation (still using the basis $v_1, \ldots, v_m$ concentrated in degree one). Define

$$\alpha : V \to V \otimes V \subset \mathcal{L}(V) \tag{30}$$

by $v_i \mapsto v_i \otimes v_i$ and extend by linearity. By Lemma 2.4, α extends to

$$d : \mathcal{L}(V) \rightarrow \mathcal{L}(V). \tag{31}$$

Though the constructions have so far taken place over a field of characteristic zero, one might wish to make similar constructions over $\mathbb{Z}$. That is, we could consider algebras and Lie algebras over $\mathbb{Z}$ rather than over a field. Notice that if we were to work over $\mathbb{Z}$ we would be required to use $2\alpha : \mathcal{L}(V) \rightarrow \mathcal{L}(V)$ since in the free Lie algebra over $\mathbb{Z}$ $[v_i, v_i] = 2v_i \otimes v_i$.

To simplify notation, identify the tensor algebra with the free non-commutative algebra, so we write vw and v^2 in place of $v \otimes w$ and $v \otimes v$ respectively.

Let us compute d on basis elements. By (30) and (26), we see that

$$
\begin{aligned}
d(v_{i_1} v_{i_2} \ldots v_{i_n}) &= d(v_{i_1}) v_{i_2} \ldots v_{i_n} - v_{i_1} d(v_{i_2} v_{i_3} \ldots v_{i_n}) \\
&= d(v_{i_1}) v_{i_2} \ldots v_{i_n} - v_{i_1} d(v_{i_2}) v_{i_3} \ldots v_{i_n} + v_{i_1} v_{i_2} d(v_{i_3} \ldots v_{i_n}) \\
&= d(v_{i_1}) v_{i_2} \ldots v_{i_n} - v_{i_1} d(v_{i_2}) v_{i_3} \ldots v_{i_n} + \ldots + \\
&\quad (-1)^{(n-1)} v_{i_1} v_{i_2} \ldots v_{i_{n-1}} d(v_{i_n}) \\
&= v_{i_1}^2 v_{i_2} \ldots v_{i_n} - v_{i_1} v_{i_2}^2 v_{i_3} \ldots v_{i_n} + \ldots \\
&\quad + (-1)^{(n-1)} v_{i_1} v_{i_2} \ldots v_{i_{n-1}} v_{i_n}^2
\end{aligned}
\tag{32}
$$

Notice that by construction, the map of (32) is a derivation of Lie algebras. Thus $d : \mathcal{L}(V) \rightarrow \mathcal{L}(V)$ is a Lie algebra derivation of $\mathcal{L}(V)$ of degree one and $D : \mathcal{L}(V) \rightarrow \mathcal{L}(V)$ is a Lie algebra derivation of $\mathcal{L}(V)$ of degree zero.

The free Lie algebra $\mathcal{L}(V)$ has bracket

$$[,] : L_p(V) \otimes L_q(V) \rightarrow L_{p+q}(V)$$

which satisfies the usual graded Lie algebra relations:

i) (graded antisymmetry)

$$[x, y] = (-1)^{|x||y|+1}[y, x] \tag{33}$$

ii) (graded Jacobi)

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] \tag{34}$$

or equivalently

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|)}[z, [x, y]] + (-1)^{|x||y|}[y, [z, x]] = 0 \tag{35}$$

On elements of the form $v_{i_1} \otimes \ldots \otimes v_{i_p} \in V^{\otimes p}$, $v_{j_1} \otimes \ldots \otimes v_{j_q} \in V^{\otimes q}$, it is a computational exercise to show that the bracket

$$[,] : V^{\otimes p} \otimes V^{\otimes q} \rightarrow V^{\otimes p+q}, \tag{36}$$

$$
\begin{aligned}
(v_{i_1} \otimes \ldots \otimes v_{i_p}) \otimes (v_{j_1} \otimes \ldots \otimes v_{j_q}) &\mapsto v_{i_1} \otimes \ldots \otimes v_{i_p} \otimes v_{j_1} \otimes \ldots \otimes v_{j_q} \\
&\quad - (-1)^{pq} v_{j_1} \otimes \ldots \otimes v_{j_q} \otimes v_{i_1} \otimes \ldots \otimes v_{i_p}
\end{aligned}
\tag{37}
$$

may be expressed in terms of the group ring $\mathbb{Z}[\Sigma_{p+q}]$ as right multiplication by

$$B_{p,q} = (1 - (-1)^{pq} \sigma^p_{(p+q, p+q-1, \ldots, 2, 1)}). \tag{38}$$

Suppose $a \in L_p$ and $b \in L_q$. Then a (respectively b) is a linear combination of elements of the form $R_{w_p}(v_{i_1} \otimes \ldots \otimes v_{i_p})$ (respectively $R_{w_q}(v_{j_1} \otimes \ldots \otimes v_{j_q})$). Since the bracket maps $L_p \otimes L_q$ into L_{p+q}, we have that

$$R_{B_{p,q}}(R_{w_p}(v_{i_1} \otimes \ldots \otimes v_{i_p}) \otimes R_{w_q}(v_{j_1} \otimes \ldots \otimes v_{j_q})) \in L_{p+q}, \tag{39}$$

which means the expression of (39) can be written as a linear combination of elements of the form $R_{w_{p+q}}(v_{k_1} \otimes \ldots \otimes v_{k_{p+q}})$.

Summarizing, we have seen that if S is interpreted as a basis concentrated in degree one, then for each $n, p, q \in \mathbb{N}$ we have the following maps:

i)

$$R_{w_n} : \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([n], S) \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([n], S) \tag{40}$$

which yielded the derivation of degree 0.

ii)

$$d : \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([n], S) \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([n+1], S) \tag{41}$$

which gave a derivation of degree 1.

iii)

$$R_{B_{p,q}} : \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([p] \amalg [q], S) \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([p] \amalg [q], S) \tag{42}$$

which, after identifying $\mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([p], S) \otimes \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([q], S)$ with $\mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([p] \amalg [q], S)$ gave the bracket of the free Lie algebra on $\mathbb{Z}[S] = \mathrm{Hom}_{\mathcal{D}_{\mathrm{Fin}}}([1], S)$.

3. Construction of the d.g.l.

This section will develop the identities, expressed as commutative diagrams, in $\mathcal{D} = \mathcal{D}_{\mathrm{Fin}}$ which allow us to construct a d.g.l.

DEFINITION 3.1. *A **complex** in $\mathcal{D}$ is a collection of objects and morphisms in $\mathcal{D}$, (A_n, d_n), indexed by $\mathbb{Z}$, such that $d_n \in \mathrm{Hom}_{\mathcal{D}}(A_n, A_{n-1})$ and $d_n \circ d_{n-1} = 0 \in \mathrm{Hom}_D(A_n, A_{n-2})$. The integer, n, corresponding to A_n may be called the degree of A_n.*

Given $S, T \in \mathrm{Fin}$ then $\mathrm{Hom}_{\mathrm{Fin}}(T, S) \in \mathrm{Fin}$. If $f \in \mathrm{Hom}_{\mathcal{D}}(T, T')$ then $f^* : \mathrm{Hom}_{\mathrm{Fin}}(T', S) \to \mathrm{Hom}_{\mathrm{Fin}}(T, S)$ is a morphism in $\mathcal{D}$ since it is a linear combination of set maps between $\mathrm{Hom}_{\mathrm{Fin}}(T, S)$ and $\mathrm{Hom}_{\mathrm{Fin}}(T', S)$.

For $Y \in \mathcal{D}$, define $\mathrm{End}_{\mathcal{D}}(Y) = \mathrm{Hom}_{\mathcal{D}}(Y, Y)$. We can define the action of a group G on an object $Y \in \mathcal{D}$ as a ring homomorphism $\mathbb{Z}[G] \to \mathrm{End}_{\mathcal{D}}(Y)$. In our case Σ_n acts on $[n] \in \mathcal{D}$, thus for any $S \in \mathrm{Fin}$, Σ_n acts on $\mathrm{Hom}_{\mathrm{Fin}}([n], S)$, so we have a homomorphism $\mathbb{Z}[\Sigma_n] \to \mathrm{End}_{\mathcal{D}}(\mathrm{Hom}_{\mathrm{Fin}}([n], S))$. With this homomorphism, $w_n \in \mathbb{Z}[\Sigma_n]$ gives an element of $\mathrm{End}_{\mathcal{D}}(\mathrm{Hom}_{\mathrm{Fin}}([n], S))$.

3.1 The "geometric" differential on the level of finite sets.

First we develop the idea of how the geometric cobar complex may be considered as arising from a diagram in $\mathcal{D}$.

For each $i \in \{1, \ldots, n+1\}$, let $\delta_i \in \mathrm{Hom}_{\mathrm{Fin}}([n+1], [n])$ be given by

$$\delta_i(j) = \begin{cases} j - 1 & \text{if } i \neq j \\ j & \text{if } i = j \end{cases} \tag{43}$$

Let $f_n \in \mathrm{Hom}_{\mathcal{D}}([n+1], [n])$ be the map given by

$$f_n = \sum_{i=1}^{n} (-1)^{i-1} \delta_i. \tag{44}$$

It is the alternating sum of the δ_i, and as a linear combination of maps in Fin, it is a map in $\mathcal{D}$.

Let $G : \mathrm{Fin}^{\mathrm{opp}} \to \mathcal{C}$ be any of the four functors, $S(\bullet)$, $\mathrm{Hom}_{\mathcal{D}}(\bullet, S)$, $X^{\bullet}$, or $F(\bullet)$ from $\mathrm{Fin}^{\mathrm{opp}}$ to

Fin, vector spaces over k, pairs, or $\mathcal{A}$. We consider each case of applying G to the complex in $\mathcal{D}$:

$$[1] \xleftarrow{f_1} [2] \xleftarrow{f_2} [3] \xleftarrow{f_3} \ldots \tag{45}$$

and for fixed $N \in \mathbb{N}$, the complex

$$[1] \xleftarrow{f_1} [2] \xleftarrow{f_2} [3] \xleftarrow{f_3} \ldots \xleftarrow{f_{N-1}} [N]. \tag{46}$$

Now G can be applied to (44). Using simplicial identities which arise from (43), one can show directly that $f_{n+1} \circ f_n = 0$. Alternatively, the following introduces the technique of proof that will follow.

PROPOSITION 3.2. $f_n \circ f_{n+1} = 0$ in the category $\mathcal{D}$.

Proof. For $S \in$ Fin, apply the functor $\operatorname{Hom}_{\mathcal{D}}(\bullet, S)$. Then

$$\operatorname{Hom}_{\mathcal{D}}([n], S) \xrightarrow{\operatorname{Hom}_{\mathcal{D}}(f_n, S)} \operatorname{Hom}_{\mathcal{D}}([n+1], S) \xrightarrow{\operatorname{Hom}_{\mathcal{D}}(f_{n+1}, S)} \operatorname{Hom}_{\mathcal{D}}([n+2], S). \tag{47}$$

After identifying $\operatorname{Hom}_{\mathcal{D}}([n], S)$ with $\mathbb{Z}[S]^{\otimes n}$ we have

$$\mathbb{Z}[S]^{\otimes n} \xrightarrow{d_n} \mathbb{Z}[S]^{\otimes n+1} \xrightarrow{d_{n+1}} \mathbb{Z}[S]^{\otimes n+2} \tag{48}$$

where d_n is the algebra derivation of (32) (where $\oplus_n \mathbb{Z}[S]^{\otimes n}$ is the algebra). Then the computation of $d \circ d$ on generators $s_i \in S$,

$$d \circ d(s_i) = d(s_i^2) = (ds_i)s_i + (-1)^{1 \cdot 1} s_i \cdot (ds_i) = s_i^3 - s_i^3 = 0 \tag{49}$$

shows that $d^2 = 0$.

Since this holds for all $S \in$ Fin, $f_n \circ f_{n+1}$ gives a natural transformation of functors

$$\operatorname{Hom}_{\mathcal{D}}([n], \bullet) \to \operatorname{Hom}_{\mathcal{D}}([n+2], \bullet) \tag{50}$$

which is now seen to be identically 0. Now applying (50) to the object $[n] \in$ Fin, and taking the image of the identity morphism in $\operatorname{Hom}_{\mathcal{D}}([n], [n])$ we see that $f_n \circ f_{n+1} = 0$. $\square$

In the above proof we are writing out in detail the proof of a special case of the Corollary to Yoneda's Lemma (see [Mac] Chapter 3), that every natural transformation between two representable functors $\operatorname{Hom}_{\mathcal{C}}(X, \bullet)$ and $\operatorname{Hom}_{\mathcal{C}}(Y, \bullet)$ is of the form $\operatorname{Hom}_{\mathcal{C}}(h, \bullet)$ for some $h : Y \to X$. Thus since the natural transformation in the proof is given by $\operatorname{Hom}_{\mathcal{D}}(f_n \circ f_{n+1}, \bullet)$, and the functor is zero, we conclude that $f_n \circ f_{n+1} = 0$.

In the case where we take $G = X^\bullet$ we have a version of the geometric cobar construction. Since $f_{n+1} \circ f_n = 0$, we have $G(f_n) \circ G(f_{n+1}) = 0$, and we have a complex in each category. The case of primary interest is of course when $G = F$ and we have

$$F([1]) \xrightarrow{Ff_1} F([2]) \xrightarrow{Ff_2} \ldots \tag{51}$$

and

$$F([1]) \xrightarrow{Ff_1} F([2]) \xrightarrow{Ff_2} \ldots \xrightarrow{Ff_{N-1}} F([N]) \tag{52}$$

which are objects of $\operatorname{Ch}(\mathcal{A})$, chain complexes in $\mathcal{A}$. We will abbreviate $F(n)$ for $F([n])$.

We may assemble (51) and define $R_F = \oplus_{n=1}^{\infty} F(n)$ and from (52) define $R_F^N = \oplus_{n=1}^{N} F(n)$. (We abbreviate R_F and R_F^N by R and R^N when F is understood.) The tensor product in $\mathcal{A}$ gives each of these a product structure, where in the case of (52), if $p + q > N$ then $F(p) \times F(q) \xrightarrow{\otimes} F(p+q) \equiv 0$ is defined to be zero.

The tensor product structure is associative by axiom (13), thus making R and R^N associative algebras.

Now to see that the tensor product makes (51) and (52) differential graded algebras, we must show that

$$R \times R \xrightarrow{\otimes} R \quad (\text{or } R^N \times R^N \xrightarrow{\otimes} R^N)$$

preserves the differential. That is for each $p, q \in \mathbb{N}$,

$$
\begin{array}{ccc}
F(p) \otimes F(q) & \longrightarrow & F(p+q) \\
\downarrow & & \downarrow \\
F(p+1) \otimes F(q) \oplus F(p) \otimes F(q+1) & \longrightarrow & F(p+q+1)
\end{array}
\tag{53}
$$

commutes, where the top map is given by $N(p, q)$, the left map by $(Ff_p \otimes 1, (-1)^p 1 \otimes Ff_q)$, the right map by Ff_{p+q}, and the bottom map by $N(p+1, q) + N(p, q+1)$.

PROPOSITION 3.3. R (or R^N) is a d.g.a.

Proof. We must show that the two maps of (53) are equal. $F(p) \otimes F(q) \in \mathcal{A}$ is the functor $\otimes \circ F \times F$ applied to $[p] \times [q] \in \text{Fin} \times \text{Fin}$. To obtain our top right map we apply $N(p, q)$ and the map $F(f_{p+q})$:

$$F(p) \otimes F(q) \xrightarrow{N} F(p \amalg q) \xrightarrow{F(f_{p+q})} F(p+q+1) \tag{54}$$

The following commutes by definition of natural transformation.

$$
\begin{array}{ccc}
F(p) \otimes F(q) & \xrightarrow{\ N\ } & F(p \amalg q) \\
{\scriptstyle F(f_p) \otimes 1} \downarrow & & \downarrow {\scriptstyle F(f_p \times \text{Id}_q)} \\
F(p+1) \otimes F(q) & \xrightarrow{\ N\ } & F(p+q+1)
\end{array}
\tag{55}
$$

Therefore we may replace the maps in (53) going around the bottom left by the sum of the two maps

$$F(p) \otimes F(q) \xrightarrow{N} F(p \amalg q) \xrightarrow{F(f_p \amalg \text{Id}_q)} F(p+q+1) \tag{56}$$

$$F(p) \otimes F(q) \xrightarrow{N} F(p \amalg q) \xrightarrow{F((-1)^q \text{Id}_p \amalg f_q)} F(p+q+1) \tag{57}$$

We need to prove that $(F(f_p \amalg \text{Id}_q) + F((-1)^q \text{Id}_p \amalg f_q)) \circ N = F(f_{p+q}) \circ N$. This will follow if we show that

$$f_p \amalg \text{Id}_q + (-1)^p \text{Id}_p \amalg f_q = f_{p+q} \in \text{Hom}_{\mathcal{D}}([p+q+1], [p+q]). \tag{58}$$

To prove (58), for fixed $p, q \in \mathbb{N}$, consider the two representable bi-functors, $\otimes \circ (\text{Hom}_{\mathcal{D}}([p], S) \times \text{Hom}_{\mathcal{D}}([q], T))$ and $\text{Hom}_{\mathcal{D}}([p+q+1], S \amalg T)$, for $S, T \in \text{Fin}$. These are functors from $\text{Fin} \times \text{Fin}$ to $\mathbb{Z}$–modules. Furthermore, $\otimes \circ (\text{Hom}_{\mathcal{D}}([p], S) \times \text{Hom}_{\mathcal{D}}([q], T))$ can be identified with the functor $\text{Hom}_{\mathcal{D}}([p] \amalg [q], S \amalg T)$ (since they are identified as Abelian groups). Consider the natural transformation of functors given by $\text{Hom}_{\mathcal{D}}(f_p \amalg \text{Id}_q + (-1)^p \text{Id}_p \amalg f_q - f_{p+q}, \bullet)$. We will show that this transformation is the zero map for all S and T. Taking $S = [p], T = [q]$ with $\text{Id}_p \amalg \text{Id}_q \in \text{Hom}_{\mathcal{D}}([p] \amalg [q], [p] \amalg [q])$ proves (58).

After identifying the functor $\text{Hom}_{\mathcal{D}}([n], S)$ with $\mathbb{Z}[S]^{\otimes n}$, to show that the natural transformation above is zero amounts to checking that, for all $S, T \in \text{Fin}$, the diagram

$$
\begin{array}{ccc}
\mathbb{Z}[S]^{\otimes p} \otimes \mathbb{Z}[T]^{\otimes q} & \longrightarrow & \mathbb{Z}[S \amalg T]^{\otimes p+q} \\
\downarrow & & \downarrow \\
\mathbb{Z}[S]^{\otimes p+1} \otimes \mathbb{Z}[T]^{\otimes q} \oplus \mathbb{Z}[S]^{\otimes p} \otimes \mathbb{Z}[T]^{\otimes q+1} & \longrightarrow & \mathbb{Z}[S \amalg T]^{\otimes p+q+1}
\end{array}
\tag{59}
$$

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commutes. This is a straightforward calculation using the definition of the derivation of (32). $\square$

As differential graded algebras R and R^N acquire the structure of differential graded Lie algebras in their own right, just as any associative algebra may be considered as a Lie algebra. Now R is the universal enveloping algebra of a primitive Lie algebra, which we will construct below.

We wish to produce a Lie subalgebra of R, and will make use of our work in Section 2.4 to do so.

We saw that for any $n \in \mathbb{N}$, $S \in \mathrm{Fin}$ we have a map

$$R_{w_n} : \mathrm{Hom}_{\mathcal{D}}([n], S) \to \mathrm{Hom}_{\mathcal{D}}([n], S) \tag{60}$$

which is given by right multiplication by w_n. As done before, this can be seen as a natural transformation of representable functors, and so R_{w_n} is represented as $\mathrm{Hom}_{\mathcal{D}}(f, S)$ for some morphism f in $\mathcal{D}$. Denote this morphism by $\widetilde{w_n}$. Again one may wish to think of this as the image of the identity map in the case of $S = [n]$.

So $\widetilde{w_n} : [n] \to [n]$, and as a morphism in $\mathcal{D}$, applying the functor $\mathrm{Hom}_{\mathcal{D}}([n], \bullet)$, we get the map R_{w_n}. Furthermore, from Proposition 2.2, we can conclude that $\widetilde{w_n}^2 = n\widetilde{w_n}$.

Now take $G = F$ and $\mathcal{C} = \mathcal{A}$. At this point we assume that $\mathcal{A}$ has the following property:

DEFINITION 3.4. *An additive category $\mathcal{A}$ is $\mathbb{Q}$-Karoubian if, for any morphism $f : M \to M$ in $\mathcal{A}$ such that there is an integer n such that $f \circ f = n \cdot f$ the image of f is an object in $\mathcal{A}$.*

Notice that if $\mathcal{A}$ is Abelian this condition is satisfied. If $\mathcal{A}$ is Karoubian then $f^2 = f$ implies the image of f is in $\mathcal{A}$, so the Karoubian requirement is not quite enough.

For each n we have the right multiplication by $F(\widetilde{w_n})$ on $F([n])$. By Proposition 2.2, the $\mathbb{Q}$-Karoubian hypothesis implies that the image of $F(\widetilde{w_n})$ is an object of $\mathcal{A}$. For the functor $\mathrm{Hom}_{\mathcal{D}}([n], \bullet)$ there is also no problem in taking images, though we will need to invert 2 to get all our desired properties. However, when working with the functor $X^{\bullet}$, we cannot take the image of $X(\widetilde{w_n})$.

The rest of this section is dedicated to proving the following theorem:

THEOREM 3.5. *Let $\mathcal{P}_F = \oplus_{n=1}^{\infty} F(n)F(\widetilde{w_n})$ and let $\mathcal{P}_F^N = \oplus_{n=1}^{N} F(n)F(\widetilde{w_n})$. $\mathcal{P}_F$ and $\mathcal{P}_F^N$ are differential graded Lie algebras in $\mathcal{A}$.*

3.2 Closure under the differential

We would first like to show $\mathcal{P}_F$ and $\mathcal{P}_F^N$ are closed under the differential $F(f_n)$:

To show that $\mathcal{P}_F$ and $\mathcal{P}_F^N$ are subcomplexes we must show that we have a map

$$F(f_n) : F(n)F(\widetilde{w_n}) \to F(n+1)F(\widetilde{w_{n+1}}). \tag{61}$$

That is, since we already know $F(f_n) \circ F(f_{n+1}) = 0$, we are required to show that

$$F(f_n)(F(n)F(\widetilde{w_n})) \subset F(n+1)F(\widetilde{w_{n+1}}). \tag{62}$$

If we show that there exists ϕ_n such that

$$\begin{array}{ccc} [n+1] & \xrightarrow{f_n} & [n] \\ {\scriptstyle \widetilde{w_{n+1}}} \downarrow & & \downarrow {\scriptstyle \widetilde{w_n}} \\ [n+1] & \xrightarrow{\phi_n} & [n] \end{array} \tag{63}$$

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commutes in $\mathcal{D}$, then applying F, we have a commutative diagram

$$
\begin{array}{ccc}
F(n) & \xrightarrow{F(\phi_n)} & F(n+1) \\
{\scriptstyle F(\widetilde{w_n})}\downarrow & & \downarrow{\scriptstyle F(\widetilde{w_{n+1}})} \\
F(n) & \xrightarrow{F(f_n)} & F(n+1)
\end{array}
\tag{64}
$$

which, in particular, shows that (62) holds. It is important to note that ϕ_n is *not* the same as f_n.

The following proposition is the first step in showing (63) commutes.

PROPOSITION 3.6. *There exists $g_n \in \mathcal{D}$ such that the following diagram commutes for all $S \in \mathrm{Fin}$*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}([n], S) & \xrightarrow{g_n} & \mathrm{Hom}_{\mathcal{D}}([n+1], S) \\
{\scriptstyle w_n}\downarrow & & \downarrow{\scriptstyle w_{n+1}} \\
\mathrm{Hom}_{\mathcal{D}}([n], S) & \xrightarrow{f_n^*} & \mathrm{Hom}_{\mathcal{D}}([n+1], S)
\end{array}
\tag{65}
$$

REMARK 3.7. *In the course of the proof we will see that 2 needs to be inverted to make this Proposition hold.*

Proof. Suppose $S = \{s_1, \ldots, s_m\}$. (65) may be identified with

$$
\begin{array}{ccc}
\mathbb{Z}[S]^{\otimes n} & \xrightarrow{g_n} & \mathbb{Z}[S]^{\otimes n+1} \\
{\scriptstyle w_n}\downarrow & & \downarrow{\scriptstyle w_{n+1}} \\
\mathbb{Z}[S]^{\otimes n} & \xrightarrow{d} & \mathbb{Z}[S]^{\otimes n+1}
\end{array}
\tag{66}
$$

where we must solve for g_n.

Let us look at $2d$ on generators of $\mathcal{L}(\mathbb{Z}[S])$: $2d(s_i) = [s_i, s_i]$, and thus

$$
2d(\mathcal{L}(\mathbb{Z}[S])) \subset \mathcal{L}(\mathbb{Z}[S]).
$$

Then

$$
\frac{1}{2} d(R_{w_n}(s_{j_1} \otimes s_{j_2} \otimes \ldots \otimes s_{j_n}))
\tag{67}
$$

$$
= \frac{1}{2} d([s_{j_1}[s_{j_2}[\ldots [s_{j_{n-1}}, s_{j_n}] \ldots]]) \in L_{n+1}(\mathbb{Z}[S])
\tag{68}
$$

$$
= R_{w_{n+1}}(\text{ an expression in } s_j\text{'s of degree } n+1)
\tag{69}
$$

$$
= R_{w_{n+1}}(g_n(s_{j_1} \otimes s_{j_2} \otimes \ldots \otimes s_{j_n}))
\tag{70}
$$

which defines g_n. $\square$

Since Proposition 3.6 holds for all S, taking $S = [n]$ and looking at the image of the identity map under g_n, we obtain $\phi_n \in \mathrm{Hom}([n+1], [n])$. Thus we have (63), completing the proof that $\mathcal{P}_F$ and $\mathcal{P}_F^N$ are complexes.

3.3 Closure under the bracket

Now to construct a Lie algebra we need to show that taking the images defined by the $F(\widetilde{w_n})$ will be closed under the bracket.

Recall from (42) that right multiplication by $B_{p,q}$, $R_{B_{p,q}} : \mathrm{Hom}_{\mathcal{D}}([p]\amalg[q], S) \to \mathrm{Hom}_{\mathcal{D}}([p]\amalg[q], S)$ gave the bracket on $\mathcal{L}(\mathbb{Z}[S])$. Considering this as a natural transformation of functors, we again can conclude that there exists $\widetilde{B_{p,q}} \in \mathrm{Hom}_{\mathcal{D}}([p]\amalg[q], [p]\amalg[q])$, such that $B_{p,q} = \mathrm{Hom}_{\mathcal{D}}(\widetilde{B_{p,q}}, S)$.

Since R has an algebra structure, we can define a bracket by taking the multiplication map

$$F(p) \times F(q) \xrightarrow{\otimes} F(p) \otimes F(q) \xrightarrow{N(p,q)} F(p+q) \tag{71}$$

and adding the map

$$F(p) \times F(q) \xrightarrow{\otimes} F(p) \otimes F(q) \xrightarrow{-(-1)^{pq}\sigma} F(q) \otimes F(p) \xrightarrow{N(q,p)} F(p+q). \tag{72}$$

Note that as a map $[,] : F(p) \otimes F(q) \to F(p+q)$, $[,] = N(p,q) - (-1)^{pq}\sigma N(q,p)$. Our goal is to show that this map sends $F(p)F(\widetilde{w_p}) \otimes F(q)F(\widetilde{w_q})$ into $F(p+q)F(\widetilde{w_{p+q}})$.

By the definition of $\widetilde{B_{p,q}}$ (38) and the commutativity axiom condition on F, we have that the following diagram commutes:

$$\begin{array}{ccc}
F(p) \otimes F(q) & \xrightarrow{N(p,q)} & F(p+q) \\
& \searrow{\scriptstyle [,]} & \downarrow{\scriptstyle F(\widetilde{B_{p,q}})} \\
& & F(p+q).
\end{array} \tag{73}$$

Since N is a natural transformation, the following diagram commutes:

$$\begin{array}{ccc}
F(p) \otimes F(q) & \xrightarrow{N} & F([p] \amalg [q]) \\
\scriptstyle F(\widetilde{w_p}) \otimes F(\widetilde{w_q}) \downarrow & & \downarrow \scriptstyle F(\widetilde{w_p} \amalg \widetilde{w_q}) \\
F(p) \otimes F(q) & \xrightarrow{N} & F([p] \amalg [q]),
\end{array} \tag{74}$$

thus the image of $F(p)F(\widetilde{w_p}) \otimes F(q)F(\widetilde{w_q})$ under N is contained in the image of $F(\widetilde{w_p} \amalg \widetilde{w_q})$.

Thus if we show that for each $p, q \in \mathbb{N}$ there exists a map $\widetilde{\psi_{p,q}}$ so that the following diagram in $\mathcal{D}$ commutes:

$$\begin{array}{ccc}
[p] \amalg [q] & \xleftarrow{\widetilde{\psi_{p,q}}} & [p] \amalg [q] \\
\scriptstyle \widetilde{w_p} \amalg \widetilde{w_q} \uparrow & & \uparrow \scriptstyle \widetilde{w_{p+q}} \\
[p] \amalg [q] & \xleftarrow{\widetilde{B_{p,q}}} & [p] \amalg [q],
\end{array} \tag{75}$$

then applying F yields

$$\begin{array}{ccc}
F([p] \amalg [q]) & \xrightarrow{F(\widetilde{\psi_{p,q}})} & F([p] \amalg [q]) \\
\scriptstyle F(\widetilde{w_p} \amalg \widetilde{w_q}) \downarrow & & \downarrow \scriptstyle F(\widetilde{w_{p+q}}) \\
F([p] \amalg [q]) & \xrightarrow{F(\widetilde{B_{p,q}})} & F([p] \amalg [q]),
\end{array} \tag{76}$$

which is enough to show that the bracket maps $F(p)F(\widetilde{w_p}) \otimes F(q)F(\widetilde{w_q})$ into $F(p+q)F(\widetilde{w_{p+q}})$.

Once we have Proposition 3.8 below, we prove (75) by arguing as before, taking $S = [p] \amalg [q]$ and looking at the image of the identity map in $\mathrm{Hom}_{\mathcal{D}}([p] \amalg [q], [p] \amalg [q])$ under $\widetilde{\psi_{p,q}}$.

PROPOSITION 3.8. *There exists $\widetilde{\psi_{p,q}} \in \mathcal{D}$ such that for any $S \in \mathrm{Fin}$*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}([p] \amalg [q], S) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\widetilde{\psi_{p,q}}, S)} & \mathrm{Hom}_{\mathcal{D}}([p] \amalg [q], S) \\
\scriptstyle \mathrm{Hom}_{\mathcal{D}}(\widetilde{w_p} \amalg \widetilde{w_q}, S) \downarrow & & \downarrow \scriptstyle \mathrm{Hom}_{\mathcal{D}}(\widetilde{w_{p+q}}, S) \\
\mathrm{Hom}_{\mathcal{D}}([p] \amalg [q], S) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\widetilde{B_{p,q}}, S)} & \mathrm{Hom}_{\mathcal{D}}([p] \amalg [q], S)
\end{array} \tag{77}$$

commutes.

Proof. Suppose $S = \{s_1, \ldots, s_m\}$. (77) may be identified with the following diagram where we are trying to solve for $\psi_{p,q}$ such that

$$\begin{array}{ccc}
\mathbb{Z}[S]^{\otimes p} \otimes \mathbb{Z}[S]^{\otimes q} & \xrightarrow{\psi_{p,q}} & \mathbb{Z}[S]^{\otimes p+q} \\
{\scriptstyle R_{w_p} \otimes R_{w_q}} \downarrow & & \downarrow {\scriptstyle R_{w_{p+q}}} \\
\mathbb{Z}[S]^{\otimes p} \otimes \mathbb{Z}[S]^{\otimes q} & \xrightarrow{[,]} & \mathbb{Z}[S]^{\otimes p+q}
\end{array} \tag{78}$$

commutes. Now the image of the left map is $L_p(\mathbb{Z}[S]) \otimes L_q(\mathbb{Z}[S])$, and the bracket maps this to $L_{p+q}(\mathbb{Z}[S])$. Thus given generators $s_{i_1} \otimes s_{i_2} \otimes \ldots \otimes s_{i_p}$ and $s_{j_1} \otimes s_{j_2} \otimes \ldots \otimes s_{j_q}$ of $\mathbb{Z}[S]^{\otimes p}$ and $\mathbb{Z}[S]^{\otimes q}$, the composite of the left and bottom map yields

$$[R_{w_p}(s_{i_1} \otimes s_{i_2} \otimes \ldots \otimes s_{i_p}), R_{w_q}(s_{j_1} \otimes s_{j_2} \otimes \ldots \otimes s_{j_q})] \in L_{p+q}(\mathbb{Z}[S]) \tag{79}$$

$$= R_{w_{p+q}}(\text{ an expression in the } s_i, s_j) \tag{80}$$

$$= R_{w_{p+q}}(\psi_{p,q}) \tag{81}$$

which defines $\psi_{p,q}$. $\square$

Notice that since the differential is compatible with the algebra structure (i.e. it is a graded algebra derivation), the differential is automatically a graded Lie derivation, and thus $\mathcal{P}_F$ and $\mathcal{P}_F^N$ are differential graded Lie algebras, thus completing the proof of Theorem 3.5.

REMARK 3.9. $\mathcal{P}_F^N$ *is a Lie-quotient of* $\mathcal{P}_F$, *as the kernel is the Lie ideal generated by all "words" of bracket length greater than* N *(which is clearly stable under the bracket).*

4. Applying the Construction to the Abelian Category of Motives

Now we specialize to the category of motives. Here we summarize some of the main facts concerning Nori's work:

4.1 Summary of EHM and C_*

Fix an embedding of $k \hookrightarrow \mathbb{C}$. All varieties and morphisms discussed below will be defined over k. Let $\mathbb{Z}-$mod denote the category of all finitely generated Abelian groups, and let Ab denote the category of all Abelian groups. All these constructions are made with integer coefficients.

FACT 4.1. *EHM is an Abelian category equipped with a faithful exact functor,* $ff : \text{EHM} \to \mathbb{Z}-\text{mod}$.

FACT 4.2. *For all varieties X and closed subvarieties $Y \subset X$ (defined over k), and all non-negative integers q, there is an object $H_q(X,Y)$ of* EHM.

FACT 4.3. *With pairs (X,Y) and (X',Y') as above, every commutative diagram*

$$\begin{array}{ccc}
Y & \xrightarrow{g} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & X'
\end{array}$$

induces a morphism $H_q(f,g) : H_q(X,Y) \to H_q(X',Y')$ in EHM.

FACT 4.4. *For $Z \subset Y \subset X$ closed subvarieties of X, there are boundary operators*

$$H_q(X,Y) \to H_{q-1}(Y,Z).$$

FACT 4.5. *Let $H_q(X(\mathbb{C}), Y(\mathbb{C}))$ denote the q-th singular homology of the pair $(X(\mathbb{C}), Y(\mathbb{C}))$ with $\mathbb{Z}$ coefficients. Then*

$$ff\, H_q(X, Y) \cong H_q(X(\mathbb{C}), Y(\mathbb{C}))$$

FACT 4.6. *Applying ff to Fact 4.3 yields the arrows on singular homology induced by (f, g) and applying ff to Fact 4.4 yields the usual boundary operator.*

Because ff is faithful exact, from the standard properties of singular homology we deduce Facts 4.7 – 4.9:

FACT 4.7. *For $X \supset Y \supset Z \supset W$ (defined over k) and any $q \in \mathbb{Z}$, the composite of the boundary operators in EHM*

$$H_q(X, Y) \to H_{q-1}(Y, Z) \to H_{q-2}(Z, W)$$

is zero.

FACT 4.8. *There is the usual exact sequence of a triple (X, Y, Z) in EHM (for the usual exact sequence of a triple in the topological setting see chapter 4 section 8 of [Sp]).*

FACT 4.9. *With notation as in Fact 4.3 whenever the composite is defined we have*

$$H_q(f, g) \circ H_q(f', g') = H_q(f \circ f', g \circ g').$$

FACT 4.10. *There is a functor $\otimes : \mathrm{EHM} \times \mathrm{EHM} \to \mathrm{EHM}$ such that for all objects A, B in EHM we have*

$$ff(A \otimes B) = ff A \otimes ff B.$$

This implies that the $\otimes$ structure on EHM satisfies the usual associativity and commutativity constraints, and $H_0(\mathrm{Spec}(k), \emptyset) = \mathbf{1}$ serves as an identity.

DEFINITION 4.11. *An r-admissible pair consists of an affine variety X and a closed subvariety Y of X such that $\dim Y < r$, $X - Y$ is smooth of pure dimension r, and*

$$H_s(X(\mathbb{C}), Y(\mathbb{C})) = \begin{cases} 0 & \text{if } s \neq r \\ \text{a free Abelian group} & \text{if } s = r. \end{cases}$$

FACT 4.12. *For r_i-admissible pairs (X_i, Y_i) for $i = 1, 2$ we have*

$$H_{r_1}(X_1, Y_1) \otimes H_{r_2}(X_2, Y_2) = H_{r_1+r_2}((X_1, Y_1) \times (X_2, Y_2)).$$

Now IndEHM is the category obtained by taking direct limits of objects in EHM. (For a summary of "Ind" categories see [De].) The properties of ff imply that the objects of EHM satisfy the ascending chain condition. It follows that IndEHM is an Abelian category. We denote by ff, once again, the induced functor $ff : \mathrm{IndEHM} \to \mathrm{Ab}$.

The form of the lemma below is taken from [No]. It was first proved by Beilinson [Be].

THEOREM 4.13 THE BASIC LEMMA. *Let X be an affine variety of dimension n, and let $W \subset X$ be a closed subvariety of dimension less than n. Then there exists a closed subvariety Z, such that $X \supset Z \supset W$ and (X, Z) is an n-admissible pair.*

Let X be an affine variety of dimension n. An admissible filtration $\mathcal{F}$ of X is a filtration of X by Zariski closed sets $X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$ such that (X_i, X_{i-1}) is an i-admissible pair. An admissible filtration $\mathcal{F}$ for X defines a complex

$$H_n(X, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2}) \to \ldots \to H_0(X_1, \emptyset)$$

which we call $C_*(X, \mathcal{F})$. Observe that $H_q(ff C_*(X, \mathcal{F})) = H_q(X(\mathbb{C}))$ for the same reason that cellular homology agrees with singular homology for cell complexes. The Basic Lemma assures us that any filtration $\{X_q\}_q$ with $\dim X_q \leqslant q$ is contained in an admissible filtration. Consequently admissible filtrations form a direct system. The direct limit of $C_*(X, \mathcal{F})$ taken over all admissible filtrations is a complex in IndEHM called $C_*(X)$.

In fact $X \mapsto C_*(X)$ is a functor from affine varieties over k to chain complexes in IndEHM. One may observe that a closed embedding $Y \hookrightarrow X$ gives a monomorphism $C_*(Y) \to C_*(X)$ and we may define $C_*(X, Y) = C_*(X)/C_*(Y)$. Again the Basic Lemma is used to show that this is a functor since given $f : X \to Y$ for any filtration $\{X_q\}$ applying f and taking Zariski closures gives a filtration $\{\overline{f(X_q)}\}$ for which we can find an admissible filtration on Y which dominates it. Thus in the limit we have a map $C_*(X) \to C_*(Y)$.

Given $\mathcal{F}$ an admissible filtration on X and $\mathcal{G}$ an admissible filtration on Y, then $\mathcal{F} \times \mathcal{G}$ is an admissible filtration on $X \times Y$, where

$$(\mathcal{F} \times \mathcal{G})_m = \bigcup_{p+q=m} \mathcal{F}_p X \times \mathcal{G}_q Y.$$

It follows from Fact 4.12 that

$$C_*(X, \mathcal{F}) \otimes C_*(Y, \mathcal{G}) = C_*(X \times Y, \mathcal{F} \times \mathcal{G}).$$

Taking direct limits over all $\mathcal{F}$ and $\mathcal{G}$ we get

$$C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

in IndEHM. The tensor structure gives a functor $\mathrm{Ch}(\mathrm{IndEHM}) \times \mathrm{Ch}(\mathrm{IndEHM}) \xrightarrow{\otimes} \mathrm{Ch}(\mathrm{IndEHM})$ which satisfies the usual commutativity and associativity constraints of the tensor product – in particular there is a Σ_n-equivariant map $C_*(X)^{\otimes n} \to C_*(X^n)$ (refer to Proposition 2.1). That is, we have a natural transformation from the functor $\otimes \circ (C_* \times C_*) : \mathrm{Aff}_k \times \mathrm{Aff}_k \to \mathrm{Ch}(\mathrm{IndEHM})$ to the functor $C_* \circ \times : \mathrm{Aff}_k \times \mathrm{Aff}_k \to \mathrm{Ch}(\mathrm{IndEHM})$.

The lemma of homological algebra below yields subcomplexes

$$\mathrm{Sing}(X(\mathbb{C})) \supset P_*(X) \supset Q_*(X)$$

such that both of the arrows

$$ff C_*(X) = P_*(X)/Q_*(X) \twoheadleftarrow P_*(X) \hookrightarrow \mathrm{Sing}(X(\mathbb{C}))$$

are quasi-isomorphisms.

Furthermore, both

$$
\begin{array}{ccc}
P_*(X) \otimes P_*(Y) & \hookrightarrow & \mathrm{Sing}(X(\mathbb{C})) \otimes \mathrm{Sing}(Y(\mathbb{C})) \\
\downarrow & & \downarrow{\scriptstyle EZ} \\
P_*(X \times Y) & \hookrightarrow & \mathrm{Sing}(X(\mathbb{C}) \times Y(\mathbb{C}))
\end{array}
$$

and

$$
\begin{array}{ccc}
ff C_*(X) \otimes ff C_*(Y) & \twoheadleftarrow & P_*(X(\mathbb{C})) \otimes P_*(Y(\mathbb{C})) \\
\downarrow & & \downarrow \\
ff C_*(X \times Y) & \twoheadleftarrow & P_*(X(\mathbb{C}) \times Y(\mathbb{C}))
\end{array}
$$

commute. Since P_*, Q_* and $P_*/Q_* = ff C_*$ are all free these maps are all quasi-isomorphisms.

Now (EHM, ff) is universal for Facts $1 - 6$ and the tensor structure is determined by Facts 4.10 and 4.12. It follows that the d.g.l. constructed in section 3 is a valid construction for (EHM, ff).

That is, by taking the functor $F = C_*(X^\bullet) \otimes \mathbb{Q}$, $\mathcal{P}_F$ is a d.g.l. in $\mathcal{M}$. Since we only use the properties above, the construction is valid for mixed Hodge structures, Galois modules, etc.

Lemma 4.14 Lemma of Homological Algebra. *Suppose the following conditions are satisfied in an abelian category*

  i) (A, ∂) *is a chain complex*

  ii) $\{F_p A\}_{p \in \mathbb{Z}}$ *is an increasing filtration of subcomplexes of* A

  iii) *for all* $n \in \mathbb{Z}$, $\cup_p F_p A_n = A_n$

  iv) *for all* $n \in \mathbb{Z}$ *there exists* $p \in \mathbb{Z}$ *such that* $F_p A_n = 0$

  v) *for all* $p \neq q$ *in* $\mathbb{Z}$, $H_q(gr_p^F A) = 0$

Then define P_n *to be the kernel of the composite map*

$$F_n A_n \to F_n A_{n-1} \to gr_n^F A_{n-1}$$

and set $Q_n = \partial(F_n A_{n+1}) + F_{n-1} A_n$. *Then*

  i) $Q \subset P \subset A$ *are subcomplexes.*

  ii) *The quotient complex* P/Q *is the familiar complex*

$$\ldots \to H_{n+1}(gr_{n+1}^F A) \to H_n(gr_n^F A) \to H_{n-1}(gr_{n-1}^F A)$$

  iii) $P \to P/Q$ *and* $P \to A$ *are both quasi-isomorphisms.*

    This is a standard lemma, but as we were unable to find a reference for it, we give the following proof due to Nori.

Proof. The first two assertions are straightforward and rely only on assumptions (i) and (ii). For the third we need the acyclicity of Q and A/P. First recall that for a complex (D, ∂) and an integer r we have the quotient complex $\tau_{<r}(D)$ given by

$$\ldots 0 \to D_r/Z_r(D) \to D_{r-1} \to D_{r-2} \to \ldots$$

and the subcomplex $\tau_{>r}(D)$ given by

$$\ldots \to D_{r+2} \to D_{r+1} \to B_r(D) \to 0 \to 0 \ldots$$

where $B_r(D) = \mathrm{im}\partial(D_{r+1})$ and $Z_r(D) = \ker(\partial : D_r \to D_{r-1})$.

    Now the increasing filtration $\{F_p A\}_{p \in \mathbb{Z}}$ induces increasing filtrations $\{F_p(A/P)\}$ and $\{F_p Q\}$. We see that

$$gr_r^F(A/P) = \tau_{>r}(gr_r^F A)$$

and

$$gr_r^F(Q) = \tau_{<r}(gr_r^F A).$$

The last hypothesis shows that the complexes $gr_r^F(A/P)$ and $gr_r^F(Q)$ are acyclic. Then the five lemma shows that both $F_a Q/F_b Q$ and $F_a(A/P)/F_b(A/P)$ are acyclic whenever $a > b$. Since the third and fourth hypotheses hold for Q and A/P as well, we deduce that Q and A/P are acyclic. $\square$

    The lemma is applied above by taking $A = \mathrm{Sing}(X(\mathbb{C}))$ and

$$F_p A = \lim_{\substack{Y \hookrightarrow X, \text{ closed} \\ \dim Y \leqslant p}} \mathrm{Sing}(Y(\mathbb{C}))$$

and then $P/Q = ff C_*(X)$.

4.2 Reduction to singular chains

Since we will work with the d.g.l. $\mathcal{P}_F$ we tensor every Abelian group with $\mathbb{Q}$ in this section. Thus, for example, $C_*(\bullet)$ really stands for $C_*(\bullet) \otimes \mathbb{Q}$, and $\mathrm{Sing}(\bullet)$ really stands for $\mathrm{Sing}(\bullet) \otimes \mathbb{Q}$.

We have a diagram

$$
\begin{array}{ccc}
\mathrm{Fin} \times \mathrm{Fin} & \xrightarrow{\;\mathrm{II}\;} & \mathrm{Fin} \\
{\scriptstyle X}\Big\downarrow\Big\downarrow{\scriptstyle X} & & \Big\downarrow{\scriptstyle X} \\
\mathrm{Aff}_k \times \mathrm{Aff}_k & \xrightarrow{\;\times\;} & \mathrm{Aff}_k \\
{\scriptstyle C_*}\Big\downarrow\Big\downarrow{\scriptstyle C_*} & & \Big\downarrow{\scriptstyle C_*} \\
\mathcal{M} \times \mathcal{M} & \xrightarrow{\;\otimes\;} & \mathcal{M}
\end{array}
\tag{82}
$$

where the top square commutes and the bottom commutes up to natural transformation. That is, there is a natural transformation

$$
N : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)
\tag{83}
$$

such that $F = C_* \circ X^\bullet$ is a functor satisfying the axioms of Section 2.3. We let $\mathcal{A} = \mathcal{M}$. Since $\mathcal{M}$ is an Abelian category, all the axioms on $\mathcal{A}$ required to apply Theorem 3.5 are satisfied and the d.g.l.s $\mathcal{P}_F$ and $\mathcal{P}_F^N$ make sense.

Nori's construction of $N : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ gives an isomorphism on homology $H_*(X) \otimes H_*(Y) \cong H_*(X \times Y)$. This motivates our fifth axiom (continuing the list from Section 2.3):

5. For all $S, T \in \mathrm{Fin}$,

$$
FS \otimes FT \xrightarrow{\;N(S,T)\;} F(S \amalg T)
\tag{84}
$$

is a quasi-isomorphism

Since we want to prove that applying the forgetful functor to $\mathcal{P}_F$ computes the rational homotopy groups, we wish to show that for the purposes of computing $H_*(ff\mathcal{P}_F)$ or $H_*(ff\mathcal{P}_F^N)$, we may replace ffC_* with Sing_*.

By the discussion of the previous section, we have a commutative diagram of quasi-isomorphisms for any m, n

$$
\begin{array}{ccccc}
ffC_*(X^m) \otimes ffC_*(X^n) & \longleftarrow & P_*(X^m) \otimes P_*(X^n) & \longrightarrow & \mathrm{Sing}_*(X^m) \otimes \mathrm{Sing}_*(X^n) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
ffC_*(X^{m+n}) & \longleftarrow & P_*(X^{m+n}) & \longrightarrow & \mathrm{Sing}_*(X^{m+n})
\end{array}
\tag{85}
$$

and this diagram is compatible with the action of $\Sigma_m \times \Sigma_n$.

Now in this context $ff\mathcal{P}_F^N$ is a double complex since each term $\mathcal{P}_F(n)$ is a complex itself. If we let $F' = P_*(X^\bullet)$ and $F'' = \mathrm{Sing}_*(X^\bullet)$ then the above diagram insures that we have morphisms of double complexes

$$
ff\mathcal{P}_F^N \leftarrow \mathcal{P}_{F'}^N \to \mathcal{P}_{F''}^N
\tag{86}
$$

and we have the spectral sequence of a bounded double complex. Taking the homology from the internal differential (the one other than the "geometric" differential) the presence of quasi-isomorphisms in (85) yields an isomorphism at the E_1 term and thus isomorphisms at the E_∞ term. Thus $H_*(ff\mathcal{P}_F^N) \cong H_*(\mathcal{P}_{F''}^N)$.

Hence we define $F = \mathrm{Sing} \circ X^\bullet$ for the rest of this paper. With this definition the diagram (51) gives the geometric cobar construction. By a theorem of Adams [Ad], this complex computes the homology of the loop space of X when X is 1-connected.

Now $\mathcal{P}_F$ is a subcomplex of the geometric cobar complex R and $\mathcal{P}_F^N$ is a (split) subcomplex of R_F^N. Writing this out we have the commutative diagram:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
{\scriptstyle F(f_3)}\big\uparrow & & {\scriptstyle F(f_3)}\big\uparrow \\
\mathcal{P}_F(3) = R_{w_3}\mathrm{Sing}(X^3) \hookrightarrow & & \mathrm{Sing}(X^3) \\
{\scriptstyle F(f_2)}\big\uparrow & & {\scriptstyle F(f_2)}\big\uparrow \\
\mathcal{P}_F(2) = R_{w_2}\mathrm{Sing}(X^2) \hookrightarrow & & \mathrm{Sing}(X^2) \\
{\scriptstyle F(f_1)}\big\uparrow & & {\scriptstyle F(f_1)}\big\uparrow \\
\mathcal{P}_F(1) = \mathrm{Sing}(X) \hookrightarrow & & \mathrm{Sing}(X)
\end{array}
$$

In the case of $\mathcal{P}_F^N$ the top of this diagram would stop at

$$\mathcal{P}_F^N = R_{w_N}\mathrm{Sing}(X^N) \hookrightarrow \mathrm{Sing}(X^N)$$

5. Comparison with Rational Homotopy Theory

5.1 Basic Definitions

First we recall some facts about the Sullivan complex of polynomial differential forms with coefficients in $\mathbb{Q}$. This description is taken from [FHT].

First we recall the functor from

$$\text{Simplicial Sets} \times \text{Simplicial Cochain Algebras} \to \text{Cochain Algebras}$$

which assigns to a simplicial set K and a simplicial cochain algebra $A = \{A_n\}_{n\geqslant 0}$ the cochain complex $A(K) = \{A^p(K)\}_{p\geqslant 0}$ in the following way. (By a cochain algebra, we mean a d.g.a. of the form $A = \{A^p\}_{p\geqslant 0}$.) $A^p(K)$ is defined to be the set of simplicial morphisms from K to A^p where we define, for $\sigma \in K_n$, $\Phi, \Psi \in A^p(K)$, addition by $(\Phi + \Psi)_\sigma = \Phi_\sigma + \Psi_\sigma$, scalar multiplication by $(\lambda\Psi)_\sigma = \lambda\Psi_\sigma$ and the differential by $(d\Psi)_n = d(\Psi_n)$. The algebra structure is "point-wise" multiplication: $(\Psi \cdot \Phi)_\sigma = \Psi_\sigma\Phi_\sigma$.

Define a simplicial commutative cochain algebra $A_{PL} = \{(A_{PL})_n\}_{n\geqslant 0}$ by

$$(A_{PL})_n = \frac{\Lambda(t_0, \ldots, t_n, y_0, \ldots, y_n)}{(\sum t_i - 1, \sum y_i)}$$

where Λ denotes the free graded commutative algebra (in this case over $\mathbb{Q}$), the t_i are in degree 0, the y_i are in degree 1, and the differential is defined by $dt_i = y_i$ and $dy_i = 0$. The face and degeneracy maps are defined by considering $(A_{PL})_n$ as a sub-cochain algebra of $A_{DR}(\Delta^n)$, the de-Rham complex on Δ^n, and using the face and degeneracy maps induced from the simplices Δ^n.

Now from a topological space X we use the standard way of obtaining a simplicial set. $S_n(X)$ is defined to be the set of all singular n-simplices $\sigma : \Delta^n \to X$, where the i-th face map, $\partial_i : S_{n+1}(X) \to S_n(X)$ is given by pulling back along the face inclusions $\Delta^n \hookrightarrow \Delta^{n+1}$ and the i-th degeneracy is given by the map from Δ^{n+1} to Δ^n that collapses the j-th face. Then applying the above construction with the simplicial set $\{S_n(X)\}$ and the simplicial cochain algebra A_{PL} gives the commutative cochain algebra of polynomial rational forms on X, $A_{PL}(X)$. This is a functor from spaces to commutative cochain algebras.

If we choose a point $p \in X$ the inclusion $p \hookrightarrow X$ determines map $A_{PL}(X) \to A_{PL}(p) = \mathbb{Q}$ which makes $A_{PL}(X)$ an augmented algebra.

Let us recall some facts about the bar construction. Let (A, m, ϵ) be an augmented d.g.a. over k, $A \xrightarrow{\epsilon} k$. Let $IA = \ker(\epsilon)$, and left s denote the shift (suspension) map. Then $\mathrm{Bar}(A)$ is a coaugmented d.g.c. As a coaugmented graded coalgebra it is the tensor coalgebra

$$T(s(IA)).$$

That is, the coalgebra structure is given by

$$[a_1|\cdots|a_n] \mapsto [a_1|\cdots|a_n] \otimes 1 + \sum_{i=1}^{n-1}[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n] + 1 \otimes [a_1|\cdots|a_n].$$

The differential (which we will not write out here) can be expressed

$$d_B = d_I + d_E$$

where the internal differential, d_I, comes from the differential on A, while the external differential d_E comes from the multiplicative structure in A. $\mathrm{Bar}(A)$ may be considered a double complex by indexing with $-s$ the tensor (external) degree and with t the differential form (internal) degree,

$$B^{-s,t}(A) = [\otimes^s IA]^t. \tag{87}$$

The negative degree allows us to use the total degree $t - s$ as the cohomolgical degree of the total complex $\mathrm{Bar}(A)$.

If we *forget the differential* then $T(s(IA))$ has a product, the shuffle product,

$$sh : T(s(IA)) \otimes T(s(IA)) \to T(s(IA)).$$

For any d.g. vector space, the shuffle product exists and is compatible with the tensor coalgebra structure.

In general the shuffle product is not compatible with the differential. It is compatible when (A, m, ϵ) is graded commutative, in which case $\mathrm{Bar}(A)$ become a d.g. Hopf algebra. We will always use commutative algebras.

Forgetting the external differential, for any d.g. vector space V we can define $\widetilde{\mathrm{Bar}}(V)$ as the tensor coalgebra $T(s(V))$ with the internal differential and the shuffle product. This is also a d.g. Hopf algebra. Notice that except for the differential, $\widetilde{\mathrm{Bar}}(IA)$ and $\mathrm{Bar}(A)$ are the same, in particular they have the same tensor coalgebra structure. (This is equivalent to $\mathrm{Bar}(A)$ when A has all products equal to zero.)

Furthermore $\mathrm{Bar}(A)$ has the "external" filtration (called the "bar" filtration) defined by $\mathfrak{B}^{-s} = \oplus_{u \leqslant s} B^{-u,v}$. So $\mathbb{Q} = \mathfrak{B}^0 \subset \mathfrak{B}^{-1} \subset \ldots \subset \mathrm{Bar}(A)$. Then projection onto $\mathfrak{B}^0$ defines an augmentation which make $\mathrm{Bar}(A)$ into an augmented d.g. Hopf algebra.

Since $\mathrm{Bar}(A)$ or $\widetilde{\mathrm{Bar}}(A)$ is an augmented d.g. Hopf algebra, we can form the Lie coalgebra of indecomposables: Let J be the augmentation ideal of $\mathrm{Bar}(A)$. Then the cokernel of (shuffle) multiplication on J defines the indecomposables:

$$J \otimes J \xrightarrow{sh} J \to J/J^2 = Q\mathrm{Bar}(A) \to 0 \tag{88}$$

Recall that a space X is nilpotent if for all $p \in X$, $\pi_1(X, p)$ is a nilpotent group and the natural action of $\pi_1(X, p)$ on $\pi_i(X, p) \otimes \mathbb{Q}$ is unipotent for $i \geqslant 2$.

In [Ha] Hain defines the homotopy Lie algebra $\mathfrak{g}_*(X, p)$ to be the graded Lie algebra where $\mathfrak{g}_0$ is the $\mathbb{Q}$-Malcev completion of $\pi_1(X, p)$ and for $i \geqslant 2$ define $\mathfrak{g}_i(X, p) = \pi_{i+1}(X, p) \otimes \mathbb{Q}$ if X is nilpotent, and 0 otherwise. This definition is not necessarily compatible with the modern topology literature, so we will call this the "homotopy Lie algebra in the sense of Hain."

Now let $A(X) = A_{PL}(X)$ be the $\mathbb{Q}$–de Rham complex as in [Ha]. Hain proves that if $E^* \to \mathbb{Q}$ is an augmented commutative d.g. algebra which is quasi-isomorphic to $A(X)$ with augmentation given

by a point p, then if X is 1-connected, integration induces a natural Lie coalgebra isomorphism of the dual of the homotopy Lie algebra (in the sense of Hain) and the homology of the indecomposables of the bar construction on E^*:

$$(\mathfrak{g}_*(X,p))^\vee \cong H_*(Q\mathrm{Bar}(E^*)) \tag{89}$$

If X is not simply connected, then we only assert that

$$(\mathfrak{g}_0(X,p))^\vee \cong H_0(Q\mathrm{Bar}(E^*)). \tag{90}$$

This will serve as the foundation of our comparison theorem.

Notice with $X = (X,p)$, working with $\mathrm{Sing}(X)$,

$$0 \to \mathrm{Sing}(p) \to \mathrm{Sing}(X) \to \mathrm{Sing}(X,p) \to 0 \tag{91}$$

is dual to working with IA

$$0 \to IA \to A(X) \to A(p) \to 0. \tag{92}$$

Furthermore we will tensor with the rationals throughout, that is $\mathrm{Sing} = \mathrm{Sing} \otimes \mathbb{Q}$.

5.2 Alternate description of the bar construction

We can give another description of the bar construction from the point of view of sections 2 and 3. That is, rather than considering functors on the category $\mathcal{D}$, we recast the bar construction in terms of the functor corepresented by a commutative algebra.

For this section, let $\mathcal{A}$ be an additive $\otimes$ category satisfying the usual commutativity, associativity and unity conditions. Fix an object $A \in \mathcal{A}$ and assume that $m : A \otimes A \to A$ is a map satisfying the constraints of commutative multiplication.

Then there is a functor $A^{\otimes \bullet}$ from $\mathcal{D}$ to $\mathcal{A}$ given by

$$[n] \mapsto A^{\otimes n}$$

and on morphisms, if two elements map to the same element, the functor multiplies those terms. This is well defined since A is a commutative algebra. For example the map $[2] \to [1]$ induces multiplication $A^{\otimes 2} \xrightarrow{m} A$. On the other hand if a map in $\mathcal{D}$ misses an element in the target, the unit object of A is inserted. For example the map $[1] \to [2]$ which sends the singleton of $[1]$ to the first element of $[2]$ induces the map $A \to A^{\otimes 2}$, $a \mapsto a \otimes 1$.

Then the algebraic bar construction on A is a functor $\mathrm{Bar}^A(\bullet) : \mathcal{D} \to \mathrm{Ch}(\mathcal{A})$ obtained by applying $A^{\otimes \bullet}$ to (45). Of course Bar^A_{-N} is given by applying $A^{\otimes \bullet}$ to (46). Re-interpreting the work of Section 3 in this context we immediately get that Bar^A is a differential graded coalgebra, and thus has an underlying Lie coalgebra structure. Henceforth define $A = A(X)$, the $\mathbb{Q}$–de Rham complex, as in Section 5.1, for a fixed X. The plan of our comparison is to use the natural transformation of functors from Fin to $\mathrm{Ch}(\mathrm{Ab})$ given by integration

$$\int : (IA)^{\otimes \bullet} \to \mathrm{Sing}(X^\bullet)^\vee.$$

Integration here makes sense as we are have differential forms which can be integrated over simplices.

5.3 Comparison

Since the shuffle product is a map of degree 0, we have that the cokernel of $\mathfrak{B}^{-N} \otimes \mathfrak{B}^{-N} \xrightarrow{sh} \mathfrak{B}^{-N}$, which might be called $Q\mathfrak{B}^{-N}$ is the same as taking all terms of $Q\mathrm{Bar}(A)$ of external degree at smallest $-N$, that is

$$Q\mathfrak{B}^{-N} = \bigoplus_{-s > -N} Q\mathrm{Bar}^{-s}(A).$$

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Since for each $n \in \mathbb{N}$ integration gives a map

$$IA^{\otimes n} \xrightarrow{\int} \operatorname{Sing}(X^n)^\vee \tag{93}$$

we will show this will give a map

$$Q\mathfrak{B}^{-N} \to (\mathcal{P}_F^N \otimes \mathbb{Q})^\vee. \tag{94}$$

THEOREM 5.1. *Let F be the functor given by $ff\mathrm{C}_*(X^\bullet) \otimes \mathbb{Q}$, then $(\mathcal{P}_F^N)^\vee$ is a Lie coalgebra such that*

$$H^*(Q\mathfrak{B}^{-N}) \cong H^*((\mathcal{P}_F^N)^\vee)$$

as Lie coalgebras.

Proof. Since F satisfies the conditions of Section 4.2, the statement of the theorem reduces to the case $F = \operatorname{Sing}(X^\bullet) \otimes \mathbb{Q}$. Notice that the construction of $\mathcal{P}_F$ does not use the Alexander–Whitney comultiplication. Since the underlying coalgebra structure on the bar construction comes from the free tensor coalgebra, and the coalgebra structure on $(\mathcal{P}_F^N)^\vee$ arises from the dual of the free tensor algebra structure, their coalgebra structures are automatically compatible. In this proof, after unravelling the definition of F, we know that the image of $F(\widetilde{w_n})$, $F(n)F(\widetilde{w_n})$, is identified with $R_{w_n}\operatorname{Sing}(X^n)$. Let us denote the dual of the map $F(\widetilde{w_n}) : \operatorname{Sing}(X^n) \to \operatorname{Sing}(X^n)$ by $w_n^\vee : \operatorname{Sing}(X^n)^\vee \to \operatorname{Sing}(X^n)^\vee$ so that we may write the image of this second map as $w_n^\vee \operatorname{Sing}(X^n)^\vee$.

We need several lemmas to complete this proof:

LEMMA 5.2.

$$w_n^\vee A^{\otimes n} = (Q\mathrm{Bar}(A))^n.$$

Proof. Since the assertion is independent of the differential, we have to show that for any vector space V,

$$w_n^\vee(V^{\otimes n}) = Q\widetilde{\mathrm{Bar}}(V)^n$$

First let W be a finite dimensional vector space. By Lemma 2.3, the following sequence is exact (where w_n splits i).

$$0 \longrightarrow L_n(W^\vee) \underset{i}{\overset{w_n}{\rightleftarrows}} (W^\vee)^{\otimes n} \xrightarrow{n-w_n} (W^\vee)^{\otimes n} \tag{95}$$

where L_n refers to the degree n part of the graded free Lie algebra on $W^\vee$ concentrated in degree one. Applying $\operatorname{Hom}(-, k)$, we have the exact sequence

$$(W)^{\otimes n} \xrightarrow{(n-w_n)^\vee} (W)^{\otimes n} \underset{i^\vee}{\overset{w_n^\vee}{\rightleftarrows}} (L_n(W^\vee))^\vee \longrightarrow 0. \tag{96}$$

Dualizing the degree n term of the free Lie algebra on $W^\vee$ gives the degree n term of the free Lie coalgebra on W.

Meanwhile, $Q(\widetilde{\mathrm{Bar}}(W^\vee))^n$ is the dual of $\mathrm{Prim}(W)_n$ inside the tensor algebra $T(W)$. Considering the discussion in Section 2.4, identifying the free Lie algebra on W with the primitive Lie algebra inside the tensor algebra on W, we know that $\mathrm{Prim}(W)_n = L_n(W)$, and so the Lemma is proved for W.

Now consider any V and let $W_1 \subset W_2 \subset \ldots \subset W$ be an exhaustive filtration of finite dimensional

subspaces. For each i we have $w_n^\vee W_i^{\otimes n} = (Q\mathrm{Bar}(W_i))^n$. Furthermore, for $i < j$ we have

$$
\begin{array}{ccc}
w_n^\vee W_i^{\otimes n} & = & (Q\widetilde{\mathrm{Bar}}(W_i))^n \\
\downarrow & & \downarrow \\
w_n^\vee W_j^{\otimes n} & = & (Q\widetilde{\mathrm{Bar}}(W_j))^n
\end{array}
\tag{97}
$$

since a basis for W_i can be completed to a basis of W_j and since the relations determined by the shuffle product applied to $\widetilde{\mathrm{Bar}}(W_j)$ do not kill any generators of $(Q\widetilde{\mathrm{Bar}}(W_i))^n$. Since a vector space is the colimit of an exhaustive filtration, the Lemma follows. $\square$

LEMMA 5.3. *If* $m : M_* \to M_*$ *and* $n : N_* \to N_*$ *are morphisms of chain complexes which are idempotent, and* $f : M_* \to N_*$ *is a quasi-isomorphism which makes*

$$
\begin{array}{ccc}
M_* & \xrightarrow{f} & N_* \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
M_* & \xrightarrow{f} & N_*
\end{array}
$$

commute, then $\mathrm{im}(m) \xrightarrow{f} \mathrm{im}(n)$ *is a quasi-isomorphism.*

Proof. Since m and n are idempotent, each provides a splitting of complexes, $M \cong mM \oplus (1-m)M$ and $N \cong nN \oplus (1-n)N$. Use these isomorphisms as the vertical arrows in the commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
mM \oplus (1-m)M & \xrightarrow{f \oplus f} & nN \oplus (1-n)N
\end{array}
$$

Since homology commutes with direct sum, we conclude that $H_*(mM) \cong H_*(nN)$. $\square$

Let

$$
e_n = \frac{w_n}{n} : V^{\otimes n} \to V^{\otimes n}.
$$

Then e_n is an idempotent, $e_n^2 = e_n$.

LEMMA 5.4. *Integration,* $\int : w_n^\vee (IA)^{\otimes n} \to w_n^\vee \mathrm{Sing}(X^n)^\vee$ *is a quasi-isomorphism.*

Proof. First notice that for $\sigma \in \Sigma_n$, the following diagram commutes, and $\int$ is a quasi isomorphism.

$$
\begin{array}{ccc}
IA^{\otimes n} & \xrightarrow{\int} & \mathrm{Sing}(X^n)^\vee \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\
IA^{\otimes n} & \xrightarrow{\int} & \mathrm{Sing}(X^n)^\vee
\end{array}
\tag{98}
$$

The result now follows immediately from Lemma 5.3 by using e_n in place of w_n and observing that for vector spaces, e_n and w_n have the same image. $\square$

Now expanding out $\mathrm{Bar}(A)$ and the dual of the geometric cobar construction along their external

degrees, and integrating in each external degree we have a commutative diagram

$$IA \longleftarrow IA \otimes IA \longleftarrow IA^{\otimes 3} \longleftarrow \cdots \tag{99}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\mathrm{Sing}(X)^\vee \longleftarrow \mathrm{Sing}(X \times X)^\vee \longleftarrow \mathrm{Sing}(X^3)^\vee \longleftarrow \cdots$$

Now we wish to show that applying the $w_i^\vee$ to each term gives us a subcomplex:

LEMMA 5.5. *The following diagram commutes:*

$$w_{n+1}^\vee \mathrm{Sing}(X^{n+1})^\vee \longrightarrow w_n^\vee \mathrm{Sing}(X^n)^\vee \longrightarrow w_{n-1}^\vee \mathrm{Sing}(X^{n-1})^\vee$$

$$\int \Big\uparrow \qquad\qquad \int \Big\uparrow \qquad\qquad \int \Big\uparrow$$

$$w_{n+1}^\vee IA^{\otimes n+1} \longrightarrow w_n^\vee IA^{\otimes n} \longrightarrow w_{n-1}^\vee IA^{\otimes n-1}$$

Proof. As pointed out the the end of Section 5.2, since integration is a natural transformation of functors

$$\int : (IA)^{\otimes \bullet} \to \mathrm{Sing}(X^\bullet)^\vee$$

which respects the Σ_n action the lemma follows from (99). $\square$

Now by Lemma 5.5 we have a morphism of double complexes from $Q\mathfrak{B}^{-N}$ to $(\mathcal{P}_F^N)^\vee$ which is an isomorphism on the E_1 term of the spectral sequence of the double complex. Since both of these complexes are bounded (by $N+1$) in external degree, this implies that they have the same E_∞ term, namely $H^*(Q\mathfrak{B}^{-N}) \cong H^*((\mathcal{P}_F^N)^\vee)$. As stated before both double complexes have comultiplication from the tensor coalgebra structure, thus this isomorphism is an isomorphism of Lie coalgebras. This proves Theorem 5.1. $\square$

COROLLARY 5.6. *If X is 1-connected then*

$$\varprojlim_N H^*((\mathcal{P}_F^N)^\vee) \cong (\mathfrak{g}_*)^\vee,$$

and if X is only assumed to be connected then

$$\varprojlim_N H^0((\mathcal{P}_F^N)^\vee) \cong (\mathfrak{g}_0)^\vee$$

as Lie coalgebras and the cobracket is the dual of the Whitehead bracket.

Proof. This follows from Hain's Theorem 2.6.2 in [Ha] identifying the homology of the indecomposables of the bar construction with the dual of rational homotopy Lie algebra. $\square$

COROLLARY 5.7. *If X is of finite type (e.g. if X is an algebraic variety) and if X is 1-connected, then for fixed k there exists some $N > 0$ such that*

$$H_i(F(\mathcal{P}^N)^\vee) \xrightarrow{\cong} (\pi_{i+1}(X,p) \otimes \mathbb{Q})^\vee \qquad \forall\, i \leqslant k$$

Proof. Look at the E_1 term of the spectral sequence of the bar construction obtained by taking the homology in the internal degree. For any $q \geqslant 2$ the terms that contribute to the calculation of $\pi_q(X,p) \otimes \mathbb{Q}$ sit in total degree $q-1$, and thus are $H^q(X) \oplus H^{q+1}(X^2) \oplus \ldots \oplus H^{q+j}(X^j) \oplus \ldots$. But the Kunneth Theorem tells us that

$$H^{q+j}(X^j) = \bigoplus_{i_1+\ldots+i_j=q+j} H^{i_1}(X) \otimes \ldots \otimes H^{i_j}(X)$$

and once $j > q$ then at least one of the $i_k = 1$ and the simply connected hypothesis forces that term, and all higher terms to be zero. $\square$

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